Int. J. Solids Structures, 1973, Vol. 9, pp. 151 to 175. Pergamon Press. Printed in Great Britain

ON THE TORSIONAL STATIC STABILITY AND RESPONSE OF OPEN SECTION TUBES SUBJECTED TO THERMAL RADIATION LOADING

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Abstract—The torsional static equilibrium of structural members of open circular cross section exposed to deformation-dependent thermal loading is investigated. The equilibrium and boundary condition equations are developed. Stability criteria are established for small deformations, and equilibrium shapes are obtained for large (postbuckling) deformations. Effects of initial imperfection are introduced and solutions obtained.

The stability criteria are compared with those of previous investigations. It is shown that flaws in the theory of previous investigations lead to incorrect criteria. The present theory is shown to be in good agreement with experimental results.

NOTATION

A	initial imperfection twist rate
$\bar{A}$	constant used as parameter in function $F(\phi, \bar{A}, k^2/\gamma)$ equation (61)
B	initial rigid-body rotation with respect to thermal radiation field
$B(\phi)$	thermal "torque" function defined by equations (29) and (30) for open circular cross section (Fig. 5)
C	torsional rigidity of beam
C_1	warping rigidity of beam
C_T	constant defined in equation (40)
D	constant defined in equation (2)
E	modulus of elasticity
e	distance from geometric center to center of rotation of beam cross section (Fig. 3)
FW	notation used to denote free-to-warp boundary condition
$F(\phi, \bar{A}, k^2/\gamma)$	function defined by equation (61)
$f(\psi, \phi)$	function defined by equation (A.16)
$G(\phi)$	function defined by equation (A.1)
K	thermal conductivity
k	nondimensional rigidity ratio defined by equation (33)
l	length of beam
m	length of circumference of beam cross section
Q	thermal radiation flux intensity
Q_{cr}	value of thermal radiation flux intensity for which static instability occurs
R	radius of circular cross section
RR	notation used to denote restrained from rotation boundary condition
RW	notation used to denote restrained from warping boundary condition
$r(s)$	distance from center of rotation to tangent drawn to the middle surface of the cross section, taken positive if vector along tangent and pointing in direction of increasing s produces positive "moment" with respect to the axis of rotation (Fig. 2)
s	circumferential coordinate of beam cross section (Fig. 2)
$T(s, z)$	temperature of beam
$T_0(z)$	average temperature of beam cross section at station z
$\bar{T}(s, z)$	variation of beam temperature from average
$T_1(z)$	torque due to torsional rigidity

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$T_2(z)$	torque due to warping rigidity
$T_t(z)$	total torque in beam $[T_t(z) = T_1(z) + T_2(z)]$
$t(s)$	thickness of beam cross section (Fig. 2)
$w(s, z)$	warping of cross section with respect to plane of average warp [equation (1)]
x, y, z	rectangular Cartesian coordinates (Fig. 2)
α	coefficient of thermal expansion
γ	constant defined by equation (34)
$\delta(\theta, \phi)$	delta function (unity or zero)
ε	nondimensional e ($\varepsilon = e/R$)
ε_T	thermal emissivity of beam surface
ε_z	beam strain in z direction
ζ	nondimensional axial beam coordinate ($\zeta = z/l$)
ζ^*	value of ζ at $\phi = \phi^*$
θ	angular circumferential coordinate of beam surface (Fig. 4)
ν	Poisson's ratio for beam material
σ	Stefan–Boltzmann constant
σ_z	axial stress component (Fig. 2)
τ	shearing stress (Fig. 2)
ϕ	rotation of beam with respect to thermal flux (Fig. 4)
ϕ_0	initial imperfections in beam (difference between equilibrium position with no thermal effects and an initially straight beam with slit aligned with the flux direction)
ϕ_1	change in ϕ (from the deformed no flux condition) due to the flux, $\phi = \phi_0 + \phi_1$
ϕ^*	value of ϕ for $\partial\phi/\partial\zeta = 0$
ψ	angular circumferential coordinate of beam surface (Fig. 4)
$\omega_s(s)$	variable defined by equation (3)

INTRODUCTION

THE effects of temperature on elastic structures have been investigated for nearly 150 yr. The theory of stresses and deformations resulting from material thermal expansion (commonly denoted as thermoelastic or thermal stress analysis) is well developed [1, 2]. In general, the thermoelastic analysis begins with a thermal input or loading which is a function of the structural coordinates and time. The temperature distribution and the stresses are determined either by a coupled thermoelastic analysis or, more commonly, by an uncoupled (temperature independent of strain) analysis. In either case the thermal loading is generally assumed to be independent of the deformations of the structure. Only limited interest has been shown in thermal loadings that depend on the deformation of the structure. The early work on deformation-dependent thermal loading considered the conductive heat transfer from one body to another, for example, a hot sphere free to roll on a pair of cold rails [3]. In recent years the advent of space vehicles has generated a new interest in deformation-dependent thermal radiation loadings. These deformation-dependent loadings can lead to either static (buckling or divergence) or dynamic (divergent oscillations or flutter) instabilities [4, 5]. The deformation-dependent thermal loading and structural response are analogous to problems in aeroelasticity.

The use of slender open-section cantilever beams in artificial satellite applications [4] provides a good example of deformation-dependent thermal loading. The low torsional rigidity of such a structural member allows large rotational deformations that produce changes in the thermal solar radiation loading. The dynamic stability of these members has been discussed by a number of authors, e.g. [5–7]; however, the static stability has received little attention. This paper will investigate the torsional static equilibrium of a cantilever beam of open circular cross section exposed to thermal radiation normal to the longitudinal beam axis (Fig. 1). The thermal loading is obviously deformation-dependent since the flux distribution, relative to boom seam, on the structure is dependent on the rotation of the beam. One of the unique features of this stability problem is that no net mechanical torque can occur at any section of the beam (since one end of the beam is free and no mechanical forces are applied to the system). This differs from the buckling of a heated column that is fixed at its ends and buckles in bending [1]. In this latter problem, the instability results due to the net mechanical forces developed by the end constraints restricting the thermal expansion.

Statements regarding the stability of the problem to be considered (Fig. 1) have been made in two previous papers [4, 5]. Both conclude that buckling can occur. However,

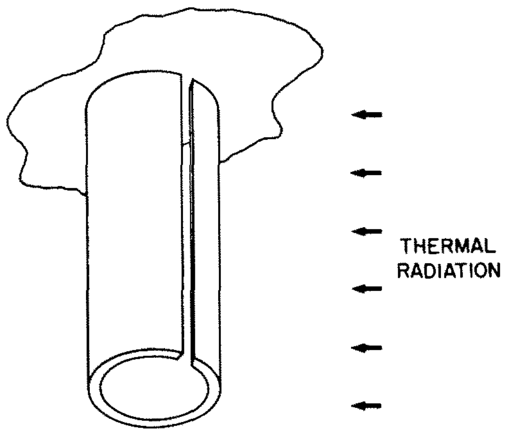


FIG. 1. Cantilever open section beam subjected to thermal radiation loading.

these analyses have flaws that lead to incorrect criteria for buckling. The first reference [4] is a static analysis that errs in the boundary condition to be applied at a free-to-warp section. This incorrect boundary condition was later used in a dynamic analysis by the same author [6]. The second incorrect result is in a dynamic analysis of a similar system [5]. This analysis was intended to determine the radiation flux flutter boundaries and succeeds in this respect. However, the results were extended to an invalid range of the mathematical model in the prediction of buckling. The model assumed that the thermal deformations were small compared to those due to the inertia loading—obviously invalid for static considerations. Jordan [7] pointed out the incorrect boundary condition used by Frisch [4] and formulated a new boundary condition which he applied qualitatively to the dynamic stability problem. However, he made no attempt to analyze the static stability.

The purpose of this paper is to correct the misconceptions presented previously by developing a valid buckling criterion and to extend the analysis to large deformations (postbuckling). Experimental results will be presented to verify the adequacy of the theory.

ANALYSIS

The theory for torsion of thin-walled members of open cross section has been well developed by Timoshenko [8] and others. The approach to the thermoelastic problem will be to extend the work of Ref. [8] to include thermal effects and deformation-dependent thermal loading. Although this is a "strength of materials approach", as opposed to a rigorous theory of elasticity analysis, the simplification will be justified by experimental results. The thermoelastic analysis will be based on an uncoupled theory (temperature independent of strains).

Equilibrium equation

Consider the twisting of an open section beam with geometry shown in Figs. 2(a) and (b). The cross section is assumed to be constant along the longitudinal axis, z. The coordinate s measures the distance along the circumference from one free edge ($s = 0$). The total length of the circumference will be denoted by m. The thickness of the section is $t(s)$. The quantity $r(s)$ is the distance from the center of rotation, C, to the tangent to the middle surface of the section drawn at point s [Fig. 2(b)]. Following the notation of Timoshenko, let $w(s, z)$ be the warping of the cross section with respect to the plane of average warp as a reference.

It has been shown in Ref. [8] that the warping of the cross section is related to the angle of twist/unit length of the bar, $\partial\phi(z)/\partial z$ [where $\phi(z)$ denotes the rotation of the cross section], by

$$w(s, z) = \frac{\partial\phi(z)}{\partial z}[D - \omega_s(s)] \tag{1}$$

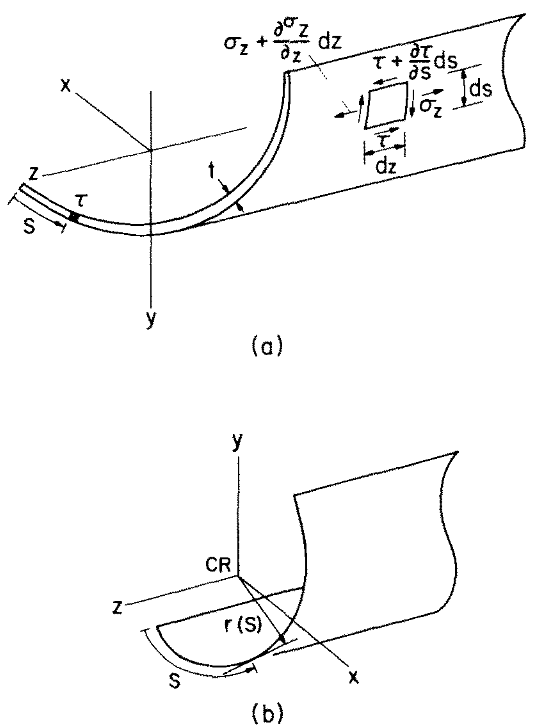


FIG. 2. Coordinate system and stress convention.

where

$$D = \frac{1}{m}\int_0^m \mathrm{d}s \int_0^s r(\bar{s})\,\mathrm{d}\bar{s} \tag{2}$$

$$\omega_s = \int_0^s r(\bar{s})\,\mathrm{d}\bar{s}. \tag{3}$$

The axial strain, $\varepsilon_z(s, z)$, is equal to $\partial w(s, z)/\partial z$; therefore,

$$\varepsilon_z(s, z) = \frac{\partial w(s, z)}{\partial z} = \phi''(z)[D - \omega_s(s)] \tag{4}$$

where primes will be used to denote derivatives with respect to the argument of a function.

The axial stress $\sigma_z(s, z)$ is related to the axial strain by

$$\varepsilon_z(s, z) = \frac{\sigma_z(s, z)}{E} + \alpha\bar{T}(s, z) \tag{5}$$

where α and E are the material coefficient of thermal expansion and the modulus of elasticity, respectively. The quantity $\bar{T}(s, z)$ denotes the temperature change relative to the average temperature $T_0(z)$. The thermal loading is thus incorporated in the stress–strain relation.

The stress displacement equation is obtained by combining equations (4) and (5):

$$\sigma_z(s, z) = E\{\phi''(z)[D - \omega_s(s)] - \alpha\bar{T}(s, z)\}. \tag{6}$$

From the equilibrium of an element of the beam [Fig. 2(b)], the relationship between the axial stress, $\sigma_z(s, z)$, and the shear stress, $\tau(s, z)$, is obtained:

$$\frac{\partial}{\partial s}[\tau(s, z)t(s)] = -t(s)\frac{\partial \sigma_z(s, z)}{\partial z}. \tag{7}$$

The shearing stress can be expressed in terms of the displacements and temperature by combining equations (6) and (7):

$$\frac{\partial}{\partial s}[\tau(s, z)t(s)] = -t(s)E\left\{\phi'''(z)[D - \omega_s(s)] - \alpha\frac{\partial \bar{T}(s, z)}{\partial z}\right\}.$$

This equation can be integrated with respect to s to obtain:

$$\tau(s, z)t(s) = -E\int_0^s \left\{\phi'''(z)[D - \omega_s(\bar{s})] - \alpha\frac{\partial \bar{T}(\bar{s}, z)}{\partial z}\right\} t(\bar{s})\,\mathrm{d}\bar{s} \tag{8}$$

where the boundary condition of zero shearing stress along the free edge of the open section $[\tau(0, z) = 0]$ has been used to evaluate the constant of integration.

The torque resulting from the warping (with temperature effects included) is the integral of the shear stress [equation (8)] times the lever arm $r(s)$ over the cross section:

$$T_2(z) = \int_0^m \tau(s, z)t(s)r(s)\,\mathrm{d}s$$

or

$$T_2(z) = -E\int_0^m \int_0^s \left\{\phi'''(z)[D - \omega_s(\bar{s})] - \alpha\frac{\partial \bar{T}(\bar{s}, z)}{\partial z}\right\} t(\bar{s})\,\mathrm{d}\bar{s}\, r(s)\,\mathrm{d}s. \tag{9}$$

If the warping rigidity C_1 is defined as

$$C_1 = E\int_0^m \int_0^s [D-\omega_s(\bar{s})]t(\bar{s})\,\mathrm{d}\bar{s}\; r(s)\,\mathrm{d}s \tag{10}$$

or after integration by parts†

$$C_1 = E\int_0^m [D-\omega_s(s)]^2 t(s)\,\mathrm{d}s \tag{11}$$

equation (9) can be written

$$T_2(z) = -C_1\phi'''(z)+\alpha E\int_0^m r(s)\int_0^s \frac{\partial \bar{T}(\bar{s}, z)}{\partial z}t(\bar{s})\,\mathrm{d}\bar{s}\,\mathrm{d}s. \tag{12}$$

The double integral in equation (12) can be reduced to a single integral in the same manner C_1 was simplified [equations (10) and (11)]; thus

$$T_2(z) = -C_1\phi'''(z)+\alpha E\int_0^m [\omega_s(m)-\omega_s(s)]\frac{\partial \bar{T}(s, z)}{\partial z}t(s)\,\mathrm{d}s. \tag{13}$$

The total torque carried by the beam is the sum of the torque due to warping and the torque $[T_1(z)]$ resulting from the torsional rigidity, C [8]:

$$T_1(z) = C\phi'(z). \tag{14}$$

The total torque, $T_t(z)$, on any section is

$$\begin{aligned} T_t(z) &= T_1(z)+T_2(z) \\ &= C\phi'(z)-C_1\phi'''(z)+\alpha E\int_0^m [\omega_s(m)-\omega_s(s)]\frac{\partial \bar{T}(s, z)}{\partial z}t(s)\,ds. \end{aligned} \tag{15}$$

The integral in the equilibrium equation (15) represents the thermal contribution to the torsion of an open cross-section beam.

Boundary conditions

If a section ($z = z_0$) of the beam is restrained from rotating (RR), then

$$\phi(z_0) = 0\;\langle RR\rangle. \tag{16}$$

† Let the integral be I and define $U(s) = \int_0^s t(\bar{s})[D-\omega_s(\bar{s})]\,\mathrm{d}\bar{s}$ and $\mathrm{d}V(s) = r(s)\,\mathrm{d}s$. It follows from equations (2) and (3) that

$$r(s)\,\mathrm{d}s = -\frac{\mathrm{d}}{\mathrm{d}s}[D-\omega_s(s)]\,\mathrm{d}s$$

therefore, after integration by parts

$$\begin{aligned} I &= UV\Big|_0^m - \int_0^m V\,\mathrm{d}U \\ &= -[D-\omega_s(s)]\int_0^s t(\bar{s})[D-\omega_s(\bar{s})]\,\mathrm{d}\bar{s}\Big|_0^m + \int_0^m [D-\omega_s(s)]t(s)[D-\omega_s(s)]\,\mathrm{d}s \end{aligned}$$

but [8] $\int_0^m [D-\omega_s(s)]t(s)\,\mathrm{d}s = 0$; therefore

$$I = \int_0^m [D-\omega_s(s)]^2 t(s)\,\mathrm{d}s.$$

If a section ($z = z_0$) is restrained from warping (RW), but not necessarily restrained from rotating, then no torque can develop from the torsional rigidity [equation (14)] and

$$\phi'(z_0) = 0 \langle RW \rangle. \tag{17}$$

At a free-to-warp section (FW), there may be applied axial stresses $\sigma_z(z_0, s)$; therefore, from equation (6)

$$\sigma_z(s, z_0) = E\{\phi''(z_0)[D-\omega_s(s)]-\alpha\overline{T}(s, z_0)\}. \tag{18}$$

Equation (18) cannot be satisfied for every value of s, but, as suggested by Vlasow [8] for the problem with no thermal effects, it can be satisfied in an average sense by multiplying each side of the equation by the weighting factor $[D-\omega_s(s)]t(s)$ and integrating from 0 to m:

$$\int_0^m \sigma_z(s, z_0)[D-\omega_s(s)]t(s)\,\mathrm{d}s$$
$$= \phi''(z_0)\int_0^m E[D-\omega_s(s)]^2 t(s)\,\mathrm{d}s - \alpha E\int_0^m [D-\omega_s(s)]\overline{T}(s, z_0)t(s)\,\mathrm{d}s.$$

If the definition of C_1 [equation (11)] is introduced, and it is recalled that warping is measured from the plane of average warping $[\int_0^m \overline{T}(z_0, s)t(s)\,\mathrm{d}s = 0]$, the boundary condition at a free-to-warp section becomes

$$\int_0^m \sigma_z(s, z_0)[D-\omega_s(s)]t(s)\,\mathrm{d}s = C_1\phi''(z_0)+\alpha E\int_0^m \omega_s(s)\overline{T}(s, z_0)t(s)\,\mathrm{d}s \ \langle FW \rangle. \tag{19}$$

If the applied stresses at the free-to-warp section are zero, then equation (19) reduces to

$$\phi''(z_0) = -\frac{\alpha E}{C_1}\int_0^m \omega_s(s)\overline{T}(s, z_0)t(s)\ \mathrm{d}s \ \langle FW \rangle, \quad \sigma_z(s, z_0) = 0. \tag{20}$$

If there are no thermal effects and no applied stresses, the boundary condition at a free-to-warp section is simply

$$\phi''(z_0) = 0 \tag{21}$$

which corresponds to the result given by Timoshenko [8].

Relative rotation of free-to-warp sections

Consider the relative rotation of two sections of a beam with constant cross-section under the following conditions: (1) the two sections are free to warp, (2) no axial stresses are applied to the sections ($\sigma_z = 0$) and (3) the only loading is the thermal loading ($T_t = 0$) with resulting thermal gradients that are continuous, well behaved functions. The equilibrium equation (15) reduces to

$$0 = C\phi'(z)-C_1\phi'''(z)+\alpha E\int_0^m [\omega_s(m)-\omega_s(s)]\frac{\partial\overline{T}(s, z)}{\partial z}t(s)\,\mathrm{d}s. \tag{22}$$

Equation (22) can be integrated once with respect to z to obtain

$$0 = C\phi(z)-C_1\phi''(z)-\alpha E\int_0^m \omega_s(s)\overline{T}(s, z)t(s)\,\mathrm{d}s+\text{const.} \tag{23}$$

where use has been made of the fact that warping is relative to the plane of average warp; therefore, $\int_0^m \bar{T}(s, z)t(s)\,ds = 0$. The boundary condition at a free-to-warp section with no applied stress ($\sigma_z = 0$) is, from equation (20),

$$\phi''(z_0) = -\frac{\alpha E}{C_1}\int_0^m \omega_s(s)\bar{T}(s, z_0)t(s)\,ds. \tag{24}$$

Equation (24) is valid at any free-to-warp section, z_0. Equation (23) is assumed valid for all z including z_0; therefore,

$$0 = C\phi(z_0) - C_1\phi''(z_0) - \alpha E\int_0^m \omega_s(s)\bar{T}(s, z_0)t(s)\,ds + \text{const.} \tag{25}$$

Equations (24) and (25) can be combined to obtain a condition at free-to-warp sections

$$\phi(z_0) = \text{const.}$$

or the rotation at each free-to-warp section is equal to the same constant. Therefore, no relative rotation can occur between two free-to-warp sections of a member within a region throughout which equation (22) and the aforementioned conditions hold. This is true no matter what temperature distribution exists on the beam, except as limited by condition (3). Note that equation (22) may not hold at a section that is held restrained from warping, so that the relative rotation of the free-to-warp sections may not be zero when a section between is held restrained from warping. The statement holds when a section between is held restrained from rotation, provided, of course, that no net torque, T_t, is caused by this restraint. Under the same conditions, the converse to the statement follows by deduction from equations (23)–(25), i.e. the free-to-warp condition, equation (24), holds at any section where the rotation is the same as at a free-to-warp section. Experiments that verify these conclusions will be discussed in a later section.

Displacement-dependent thermal loading

No specification of the temperature distribution $T(z, s)$ or the geometry of the cross section has been made. The remainder of the analysis will be focused on a particular thermal radiation loading (normal to the longitudinal axis of the beam) and a particular beam cross section (circular with uniform cross section) as shown in Fig. 3. The thermal flux magnitude, Q, and direction are assumed to remain fixed. The cross section of radius R

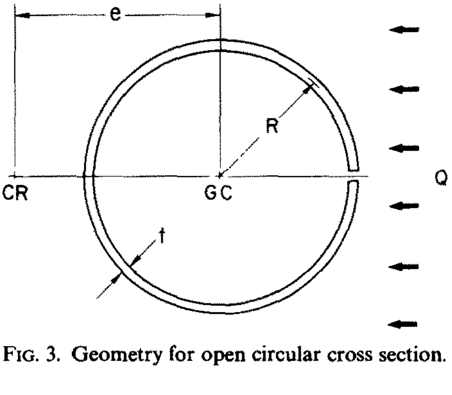


FIG. 3. Geometry for open circular cross section.

and thickness t can rotate with respect to the flux field. The thermal radiation distribution (and the resulting temperature distribution) on the beam is therefore dependent on the beam deformation.

The equilibrium temperature distribution will be obtained subject to the following assumptions:

1. The twist per unit length $[\partial\phi(z)/\partial z]$ is sufficiently small that thermal conduction in the longitudinal direction may be neglected. The temperature distribution at any section thus becomes a function of the rotation at that section, $T(s, z) = T(s, \phi(z))$.

2. The thermal conductivity, K, is sufficiently large that the variation in temperature, $\bar{T}(s, \phi(z))$, on the cross section is small when compared to the average absolute temperature, T_0;† thus

$$T(s, \phi(z)) = T_0 + \bar{T}(s, \phi(z)) \tag{26}$$

where

$$|\bar{T}| \ll T_0. \tag{27}$$

This feature is designed into many structures to reduce thermal bending deformations.

3. The amount of radiation absorbed by the beam is proportional to the cosine of the angle between the normal to the surface and the direction of the flux field. The loss of heat is by radiation from the surface of the beam.

With these basic assumptions and considerable algebraic manipulation, it can be shown (Appendix A) that the thermal loading contribution to the equilibrium equation (15) is

$$\alpha E \int_0^m [\omega_s(m) - \omega_s(s)] \frac{\partial \bar{T}(s, z)}{\partial z} t(s)\, \mathrm{d}s = -\frac{\alpha E R^5 Q}{K} B[\phi(z)]\phi'(z) \tag{28}$$

where, for $-\pi/2 \leq \phi \leq \pi/2$,

$$B(\phi) = [-\tfrac{3}{4}\pi^2 + \phi^2 + 2(\pi + \tfrac{1}{3}\pi^3)\cos\phi - 2] + \frac{e}{R}[2 - \tfrac{5}{2}\pi\cos\phi] \tag{29}$$

and for $\pi/2 \leq |\phi| \leq \pi$

$$B(\phi) = [-2 + \tfrac{1}{4}\pi^2 - 2\pi\phi + \phi^2] + \frac{e}{R}[2 - \tfrac{1}{2}\pi\cos\phi]. \tag{30}$$

The parameter e is the distance from the geometric center of the cross section to the center of rotation (Fig. 3). A plot of the function $B(\phi)$ is given in Fig. 5.

The equilibrium equation with the effects of the deformation-dependent thermal loading is

$$C\phi'(z) - C_1\phi'''(z) - \frac{\alpha E R^5 Q}{K} B[\phi(z)]\phi'(z) = 0. \tag{31}$$

Equation (31) can be nondimensionalized by the following parameters:

$$\zeta = z/l \tag{32}$$

$$k^2 = Cl^2/C_1 \tag{33}$$

† $T_0(z) = \text{const.} = T_0$ for the case being considered since the input flux is constant with z.

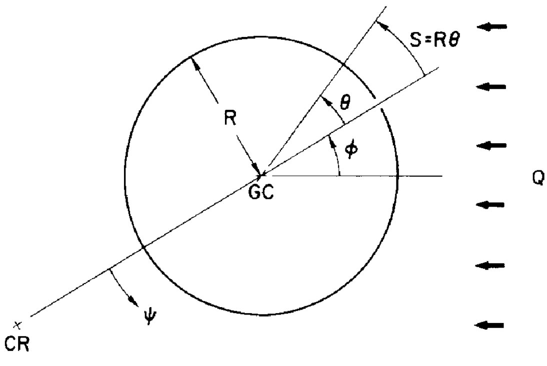


FIG. 4. Coordinate system for circular cross section.

$$\gamma = \alpha E R^5 l^2 Q / K C_1 \tag{34}$$

and

$$\varepsilon = e/R \tag{35}$$

therefore,

$$\phi'''(\zeta) - k^2\phi'(\zeta) + \gamma B(\phi)\phi'(\zeta) = 0. \tag{36}$$

For the circular cross section,

$$C = \tfrac{1}{3}\pi R t^3 \frac{E}{1+\nu} \tag{37}$$

and

$$C_1 = Et\pi R^5(\tfrac{2}{3}\pi^2 - 4\varepsilon + \varepsilon^2). \tag{38}$$

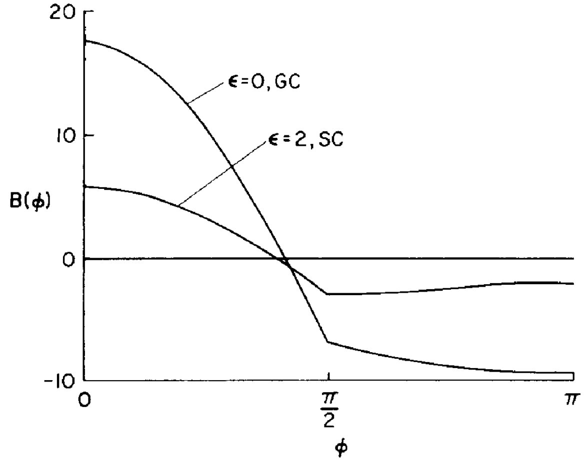


FIG. 5. The function $B(\phi)$ for geometric center and shear center.

Although a closed-form solution for the nonlinear equation (36) does not appear feasible, it is possible to perform two integrations of the equation. First, however, consider the small deflection linear stability problem.

Thermal buckling (small deflection)

Consider first small deflections $\phi(z)$ about the equilibrium condition with the "slit" in the beam straight and toward the radiation source (Fig. 3). The equilibrium equation [from equations (36) and (29)] is

$$\phi'''(\zeta)-k^2\phi'(\zeta)+\gamma\{[-\tfrac{3}{4}\pi^2+2(\pi+\tfrac{1}{3}\pi^3)-2]+\varepsilon(2-\tfrac{5}{2}\pi)\}\phi'(\zeta) = 0.$$

The thermal loading may be viewed as an equivalent mechanical torque [equation (15)]. Accordingly, the center of rotation may be taken as the shear center and for a circular cross section $\varepsilon = 2$ [8]. The equilibrium equation becomes

$$\phi'''(\zeta)-[k^2-\gamma(2-3\pi-\tfrac{3}{4}\pi^2+\tfrac{2}{3}\pi^3)]\phi'(\zeta) = 0$$

or

$$\phi'''(\zeta)+C_T\phi'(\zeta) = 0 \tag{39}$$

where

$$C_T = -[k^2-\gamma(2-3\pi-\tfrac{3}{4}\pi^2+\tfrac{2}{3}\pi^3)]. \tag{40}$$

The appropriate boundary conditions† are [equations (16), (17) and (20)]

$$\phi(\zeta_0) = 0 \ \langle RR\rangle \tag{41}$$

$$\phi'(\zeta_0) = 0 \ \langle RW\rangle \tag{42}$$

and

$$\phi''(\zeta_0) = -\gamma[2-3\pi-\tfrac{3}{4}\pi^2+\tfrac{2}{3}\pi^3]\phi(\zeta_0) \ \langle FW\rangle. \tag{43}$$

All possible combinations of boundary conditions that can be satisfied by equation (39) are shown in Table 1. The buckling condition (value of C_T for which nontrivial solutions

† The boundary condition chosen by Frisch [4, 6] for a free-to-warp section was taken erroneously to be equation (21) (which does not include the thermal effects) instead of equation (43). In the analysis by Beam [5] only cases with negligible torsional rigidity were considered ($k^2 = 0$). The boundary conditions at the built-in end ($\zeta = 0$), RR and RW, were taken as

$$\phi(0) = 0$$

$$\phi'(0) = 0.$$

The third boundary condition was taken as $\phi''(0) = 0$, which is valid if the other end of the beam, $\zeta = 1$, is free to warp (as was assumed in the analysis). This can be shown by integration of the equilibrium equation (39) ($k^2 = 0$)

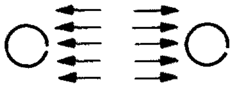
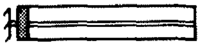
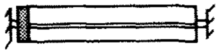
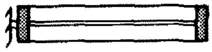
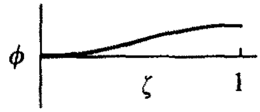
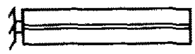
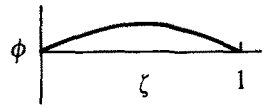
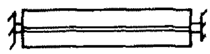
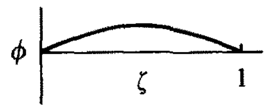
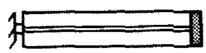
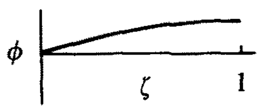
$$\phi''(\zeta)+\gamma[2-3\pi-\tfrac{3}{4}\pi^2+\tfrac{2}{3}\pi^3]\phi(\zeta) = \text{const.}$$

but at FW end $\phi''(1)+\gamma[2-3\pi-\tfrac{3}{4}\pi^2+\tfrac{2}{3}\pi^3]\phi(1) = 0$; therefore, the constant of integration is zero and

$$\phi''(\zeta)+\gamma[2-3\pi-\tfrac{3}{4}\pi^2+\tfrac{2}{3}\pi^3]\phi(\zeta) = 0$$

for all values of ζ including $\zeta = 0$. Since $\phi(0) = 0$, then $\phi''(0) = 0$ as prescribed by Beam. The error in Beam's buckling criteria came not from the boundary condition but from the application of the dynamic model (which assumed deflection from inertia loading large compared to thermal deformations) to static problems.

TABLE 1. LINEAR STATIC STABILITY CRITERIA

Case No.	Boundary conditions	Linear theory stability boundary 		Linear theory buckling mode shape $\phi(\zeta)$
1	 RR, RW FW	*	*	*
2	 RR, RW RR	*	*	*
3	 RR, RW RW	$\sqrt{C_T} = \pi$	*	
4	 RR, FW FW	$\sqrt{C_T} = \pi$	*	
5	 RR, FW RR	$\sqrt{C_T} = \pi$	*	
6	 RR, FW RW	$\sqrt{C_T} = \frac{\pi}{2}$	*	

* No static instability.

exist) and the corresponding buckling mode shape are indicated. Consider, for example, the buckling condition for cases 3–5. All three cases have the same criteria (but different buckling modes):

$$\pi = \sqrt{C_T} = \sqrt{[-k^2+\gamma(2-3\pi-\tfrac{3}{4}\pi^2+\tfrac{2}{3}\pi^3)]} \tag{44}$$

$$\pi^2 \approx -k^2+5{\cdot}84\gamma. \tag{45}$$

If the physical parameters are introduced from equations (33), (34), (37) and (38) and ν is taken as 0·3, equation (45) becomes

$$Q_{cr} = 13{\cdot}7\frac{K}{\alpha t}\left[\left(\frac{t}{l}\right)^2 + 0{\cdot}01\left(\frac{t}{R}\right)^4\right] \tag{46}$$

where Q_{cr} is the flux intensity for instability.

If a "short" beam is defined as one whose dimensions satisfy the relation

$$\left(\frac{t}{l}\right)^2 \gg 0{\cdot}01\left(\frac{t}{R}\right)^4 \quad \text{"short" beam} \tag{47}$$

and conversely for a "long" beam

$$\left(\frac{t}{l}\right)^2 \ll 0{\cdot}01\left(\frac{t}{R}\right)^4 \quad \text{"long" beam} \tag{48}$$

the buckling criteria become

$$\sqrt{\left(\frac{\alpha Q_{cr}}{Kt}\right)}\, l = 3{\cdot}70 \qquad \text{short beam} \tag{49}$$

and

$$\sqrt{\left(\frac{\alpha Q_{cr}}{Kt}\right)}\frac{R^2}{t} = 0{\cdot}370 \quad \text{long beam.} \tag{50}$$

If the thermal radiation flux is directed 180° away from the slit, the thermal torque has a change of sign as indicated in Fig. 5 for $\phi = 180°$. No instabilities occur for this loading as indicated in Table 1.

Large deformations (no initial imperfections)

Consider now solutions to the nonlinear large deformation equation (36). If $-\pi/2 \leq \phi \leq \pi/2$ then [$B(\phi)$ from equation (29) with $\varepsilon = 2$]

$$\phi''' - k^2\phi' + \gamma[(2-\tfrac{3}{4}\pi^2) + \phi^2 + (\tfrac{2}{3}\pi^3 - 3\pi)\cos\phi]\phi' = 0 \tag{51}$$

where the argument ζ has been deleted for briefness. Equation (51) can be integrated once with respect to ζ to obtain

$$\phi'' - k^2\phi + \gamma[(2-\tfrac{3}{4}\pi^2)\phi + \tfrac{1}{3}\phi^3 + (\tfrac{2}{3}\pi^3 - 3\pi)\sin\phi] = \text{const.} \tag{52}$$

For all cases in Table 1, one end is restrained from rotation

$$\phi(0) = 0\ \langle RR \rangle, \qquad \zeta = 0. \tag{53}$$

If attention is now restricted to case 5, Table 1 (a case for which small deflection buckling is possible), the end is free to warp:

$$\phi''(0) = -\gamma[(2-\tfrac{3}{4}\pi^2)\phi(0) + \tfrac{1}{3}\phi^3(0) + (\tfrac{2}{3}\pi^3 - 3\pi)\sin\phi(0)]\ \langle FW \rangle, \quad \zeta = 0. \tag{54}$$

The combination of equations (53) and (54) implies

$$\phi''(0) = 0. \tag{55}$$

Thus, from equations (52), (53) and (55), it follows that the constant of integration in equation (52) is zero; therefore,

$$\phi''-k^2\phi+\gamma[(2-\tfrac{3}{4}\pi^2)\phi+\tfrac{1}{3}\phi^3+(\tfrac{2}{3}\pi^3-3\pi)\sin\phi] = 0. \tag{56}$$

Since

$$\phi'' = \frac{\mathrm{d}\phi}{\mathrm{d}\zeta}\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\zeta}\right) \tag{57}$$

then equation (56) can be written

$$\frac{\mathrm{d}\phi}{\mathrm{d}\zeta}\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\zeta}\right) = k^2\phi-\gamma[(2-\tfrac{3}{4}\pi^2)\phi+\tfrac{1}{3}\phi^3+(\tfrac{2}{3}\pi^3-3\pi)\sin\phi] = 0 \tag{58}$$

or integrating once

$$\frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\zeta}\right)^2 = \frac{k^2}{2}\phi^2-\gamma\left[(2-\tfrac{3}{4}\pi^2)\frac{\phi^2}{2}+\tfrac{1}{12}\phi^4-(\tfrac{2}{3}\pi^3-3\pi)\cos\phi\right]+\frac{C_2}{2} \tag{59}$$

where C_2 is a constant of integration. An additional integration of equation (59) produces

$$\zeta\sqrt{\gamma} = \int_0^{\phi}\frac{\mathrm{d}\bar{\phi}}{\sqrt{[k^2/\gamma-(2-\frac{3}{4}\pi^2)]\bar{\phi}^2-\frac{1}{6}\bar{\phi}^4+2(\frac{2}{3}\pi^3-3\pi)\cos\bar{\phi}+\bar{A}}} \tag{60}$$

where $\bar{A} = C_2/\gamma$ is a constant to be determined by the third boundary condition. If the function F is defined

$$F(\phi,\bar{A},k^2/\gamma) = \int_0^{\phi}\frac{\mathrm{d}\bar{\phi}}{\sqrt{[k^2/\gamma-(2-\frac{3}{4}\pi^2)]\bar{\phi}^2-\frac{1}{6}\bar{\phi}^4+2(\frac{2}{3}\pi^3-3\pi)\cos\bar{\phi}+\bar{A}}} \tag{61}$$

then equation (60) can be abbreviated to

$$\zeta\sqrt{\gamma} = F(\phi,\bar{A},k^2/\gamma). \tag{62}$$

Equation (62) is valid for $0 \le \zeta \le \zeta^*$ where ζ^* is the value of ζ for which the radical in equation (61) is equal to zero. The corresponding value of ϕ will be denoted by ϕ^*; thus [from equation (60)]:

$$\left.\frac{\mathrm{d}\phi}{\mathrm{d}\zeta}\right|_{\substack{\zeta=\zeta^*\\ \phi=\phi^*}} = 0. \tag{63}$$

For values of $\zeta > \zeta^*$, equation (60) must be modified†

$$\zeta\sqrt{\gamma} = \left[\int_0^{\phi^*}\frac{\mathrm{d}\bar{\phi}}{\sqrt{\ }}+\int_{\phi}^{\phi^*}\frac{\mathrm{d}\bar{\phi}}{\sqrt{\ }}\right] \tag{64}$$

where the quantity under the radical has not been repeated for brevity. Equation (64) can be rewritten

$$\zeta\sqrt{\gamma} = \left[\int_0^{\phi^*}+\int_0^{\phi^*}-\int_0^{\phi}\right]$$

† The negative square root of $(\mathrm{d}\phi/\mathrm{d}\zeta)^2$ must be taken from equation (59) since $\mathrm{d}\phi/\mathrm{d}\zeta$ is negative for values of $\zeta > \zeta^*$.

or with the notation of F [equation (61)]

$$\zeta\sqrt{\gamma} = [2F(\phi^*, \bar{A}, k^2/\gamma) - F(\phi, \bar{A}, k^2/\gamma)] \quad \zeta > \zeta^*. \tag{65}$$

The constant $\bar{A}$ is determined from the boundary condition at the end $\zeta = 1$

$$\phi(1) = 0$$

therefore,

$$\sqrt{\gamma} = [2F(\phi^*, \bar{A}, k^2/\gamma)]. \tag{66}$$

Plots of rotation at the center of the beam ($\zeta = 1/2$) vs. the intensity parameter $\sqrt{\gamma}$ are shown in Fig. 6. The function F was obtained by numerical integration.

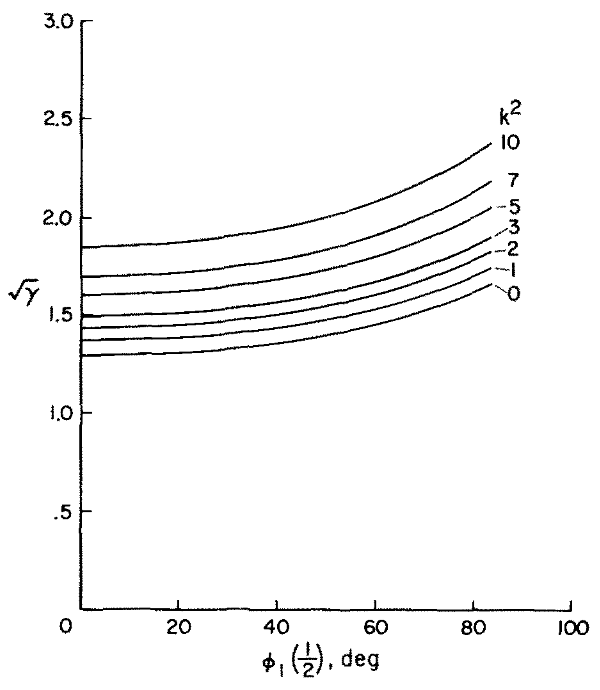


FIG. 6. Mid-length rotation for varying ratios (k^2) of torsional to warping rigidity.

Large deformations of "short" beam with initial imperfections

Consider now the effect of initial imperfections on the class of "short" beams defined by equation (47). The initial imperfections, $\phi_0(\zeta)$, are defined as the difference between the equilibrium position with no thermal effects and an initially straight beam with the slit aligned with the flux direction (Fig. 3). If $\phi_1(\zeta)$ is used to denote the change in ϕ (from the deformed no flux condition) due to the flux then the total deflection from the flux direction is

$$\phi(\zeta) = \phi_0(\zeta)+\phi_1(\zeta). \tag{67}$$

If initial linear twist A and rotation B are considered then

$$\phi(\zeta) = A\zeta+B+\phi_1(\zeta). \tag{68}$$

For short beams the torsional rigidity may be neglected in equation (51). To properly account for initial imperfections, it must be recalled that the thermal torque is dependent on the total displacement $\phi(\zeta)$ while the torque due to warping rigidity depends on the change in rotation $\phi_1(\zeta)$ from the equilibrium position. In the previous discussion there was no initial imperfection $\phi = \phi_1$; therefore, no distinction was made. Equation (51) thus becomes

$$\phi_1'''(\zeta)+\gamma[(2-\tfrac{3}{4}\pi^2)+\phi^2+(\tfrac{2}{3}\pi^3-3\pi)\cos\phi]\phi' = 0 \tag{69}$$

and the boundary conditions (case 5, Table 1)†

$$\phi_1(0) = 0 \quad \langle RR\rangle, \quad \zeta = 0 \tag{70}$$

$$\phi_1''(0) = -\gamma[(2-\tfrac{3}{4}\pi^2)\phi(0)+\tfrac{1}{3}\phi^3(0)+(\tfrac{2}{3}\pi^3-3\pi)\sin\phi(0)] \quad \langle FW\rangle, \quad \zeta = 0 \tag{71}$$

$$\phi_1(1) = 0 \quad \langle RR\rangle, \quad \zeta = 1. \tag{72}$$

For the class of imperfections being considered [equation (68)]

$$\phi''(\zeta) = \phi_1''(\zeta). \tag{73}$$

The differential equation and boundary condition [equations (69)–(72)] can be expressed in terms of ϕ

$$\phi'''(\zeta)+\gamma[(2-\tfrac{3}{4}\pi^2)+\phi^2(\zeta)+(\tfrac{2}{3}\pi^3-3\pi)\cos\phi(\zeta)]\phi'(\zeta) = 0 \tag{74}$$

$$\phi(0) = B \tag{75}$$

$$\phi''(0) = -\gamma[(2-\tfrac{3}{4}\pi^2)\phi(0)+\tfrac{1}{3}\phi^3(0)+(\tfrac{2}{3}\pi^3-3\pi)\sin\phi(0)] \tag{76}$$

$$\phi(1) = A+B. \tag{77}$$

Integration of equation (74) with respect to ζ produces

$$\phi''(\zeta)+\gamma[(2-\tfrac{3}{4}\pi^2)\phi(\zeta)+\tfrac{1}{3}\phi^3(\zeta)+(\tfrac{2}{3}\pi^3-3\pi)\sin\phi(\zeta)] = \text{const.} \tag{78}$$

Equation (78) must be valid for all ζ including $\zeta = 0$; therefore [with equation (76)], it can be concluded that the integration constant is zero. With the method used in the previous section, equation (78) can be integrated twice to obtain

$$\zeta\sqrt{\gamma} = \int_B^\phi \frac{\mathrm{d}\bar{\phi}}{\sqrt{\{[-(2-\tfrac{3}{4}\pi^2)]\bar{\phi}^2-\tfrac{1}{6}\bar{\phi}^4+2(\tfrac{2}{3}\pi^3-3\pi)\cos\bar{\phi}+\bar{A}\}}}. \tag{79}$$

With the definition of F in equation (61), equation (79) becomes

$$\zeta\sqrt{\gamma} = F(\phi,\bar{A},0)-F(B,\bar{A},0) \tag{80}$$

† Since it has been shown that no relative twist of the free-warping ends occurs in case 4, the solution obtained for case 5 will also hold for case 4.

which is valid for $\zeta < \zeta^*$. For $\zeta > \zeta^*$, equation (79) must be modified for the reason noted earlier:

$$\zeta\sqrt{\gamma} = \int_B^{\phi^*} + \int_\phi^{\phi^*} = \int_0^{\phi^*} - \int_0^B + \int_0^{\phi^*} - \int_0^\phi$$

$$\zeta\sqrt{\gamma} = 2F(\phi^*, \bar{A}, 0) - F(B, \bar{A}, 0) - F(\phi, \bar{A}, 0). \tag{81}$$

The constant $\bar{A}$ is determined by the boundary condition at $\zeta = 1$ $[\phi(1) = A+B]$; therefore

$$\sqrt{\gamma} = 2F(\phi^*, \bar{A}, 0) - F(B, \bar{A}, 0) - F(A+B, \bar{A}, 0). \tag{82}$$

The rotation that will be noted when the flux loading is applied can be obtained from equation (68) where ϕ is obtained as indicated above:

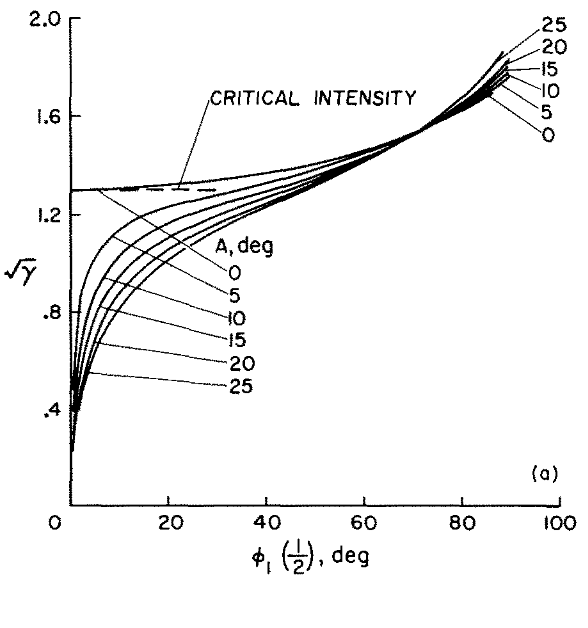
$$\phi_1(\zeta) = \phi(\zeta) - A\zeta - B. \tag{83}$$

Plots of the rotation ϕ_1 as a function of flux intensity for various initial imperfections are given in Fig. 7.

EXPERIMENT

An experimental investigation was conducted to verify the adequacy of the theory. Beams of open circular cross section with the following properties were chosen: material, Be Cu; length, $l = 0{\cdot}684$, $0{\cdot}838$, $1{\cdot}60$ m; radius, $R = 0{\cdot}00636$ m; thickness, $t = 0{\cdot}636 \times 10^{-4}$ m.

The exterior surface of the beam was painted with a flat-black paint to increase the thermal absorptivity. The thermal flux was obtained from strip lamps which, by variation of the distance to the specimen, could provide flux intensities between 0·5 and 15 solar constants. The per cent flux absorbed by the surface was determined by thermocouple



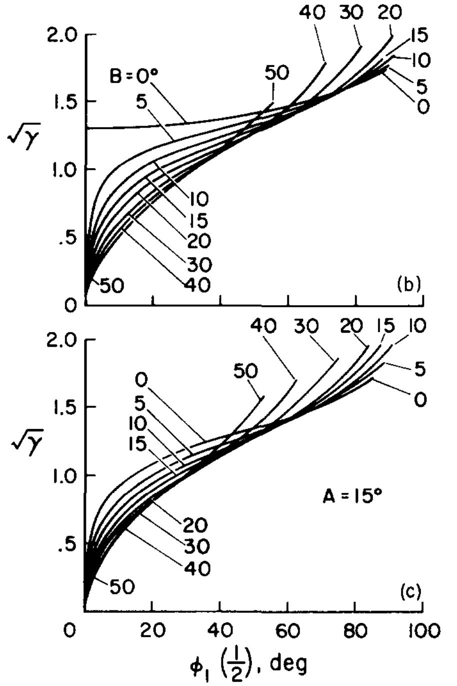


FIG. 7. (a) Mid-length rotation vs. nondimensional heat intensity for varying initial twist, no initial rigid rotation ($B = 0$); (b) mid-length rotation vs. nondimensional heat intensity for varying initial rigid rotations, initial twist $A = 0°$; (c) mid-length rotation vs. nondimensional heat intensity for varying initial rigid rotations, initial twist $A = 15°$.

measurements. Thermal conductivity of the beryllium copper beam was determined experimentally. Rigid-body imperfections, B (initial rotations), were obtained by rotating the lamps with respect to the beam axis. Initial twist imperfections, A, were obtained by selection of specimens.

Boundary conditions simulating restraint from warping (RW) were obtained by soldering the beam to a rigid cylinder that was free to rotate if no restraint from rotation was desired or restrained from rotation (RR) if desired. The restrained from rotation (RR) but free-to-warp (FW) boundary condition was simulated by restraining the beam by a small tab on the axis of symmetry of the cross section. The free-to-warp (FW) boundary conditions with no rotational constraint were, of course, left free and unrestrained.

Relative rotation of free-to-warp sections

A series of tests was conducted to verify the theoretical conclusion that the relative rotation of two free-to-warp sections must be zero for any thermal loading. Boundary conditions for case 4, Table 1, were simulated. The lamps were progressively moved around the axis of the beam ($B = 0$–2π rad.). Although large rotations of the central portion of the beam were observed for various lamp positions, no rotation of the free end ($\zeta = 1$) occurred. Since the other free-to-warp end ($\zeta = 0$) was restrained from rotation, the results substantiate the theory. For other combinations of boundary conditions (cases 1 and 6, Table 1), large (over 1 radian) relative rotations of the beam ends were observed for the same sequence of tests.

Buckling and large deformation

Quantitative measurements of the rotation at the center of the beam ($\zeta = 1/2$) were made for case 4, Table 1. The initial rigid-body rotation (B) was zero, and the initial twist was dependent on the specimen. The flux intensity was varied over the available range (0·5–15 solar constants). The pertinent data and experimental results are shown in Fig. 8. Since all the beams tested fall within the "short" classification [equation (47)], the theoretical case for comparison was obtained from the corresponding "short" beam, large-deflection theory.

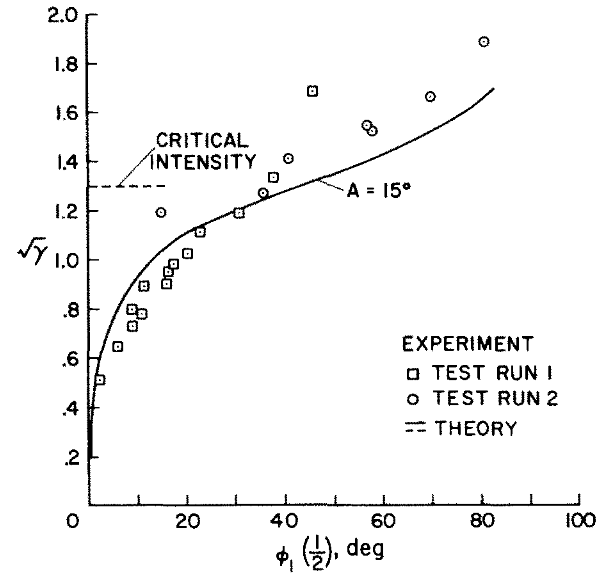


FIG. 8. Variation of mid-length rotation with nondimensional intensity.

The final series of tests consisted of a variation of the initial rigid-body rotation (B) and the measurement of the end rotation ($\zeta = 1$) for case 6, Table 1. The flux intensity (lamp distance) was held constant for this series of tests. Pertinent data and experimental results are shown in Fig. 9. Theoretical results (short beam, large deflection) are obtained by doubling the "effective length" (linear buckling load halved) and are shown for comparison. A band of 10° width is used to compare the results since error in initial alignment may be $\pm 5°$.

SUMMARY AND CONCLUDING REMARKS

Buckling criteria [equations (49) and (50)] have been developed for tubes with open circular cross section that are exposed to thermal radiation loading. As in the case of mechanically loaded beams, the torsional rigidity may be neglected for "short" beams. Definitions of "short" and "long" beams are given by equations (47) and (48). The critical heat intensity is shown to increase with thermal conductivity as expected, since a more effective heat conductor reduces the thermal loading effect. The critical heat intensity varies inversely with the coefficient of thermal expansion or, in other words, the more

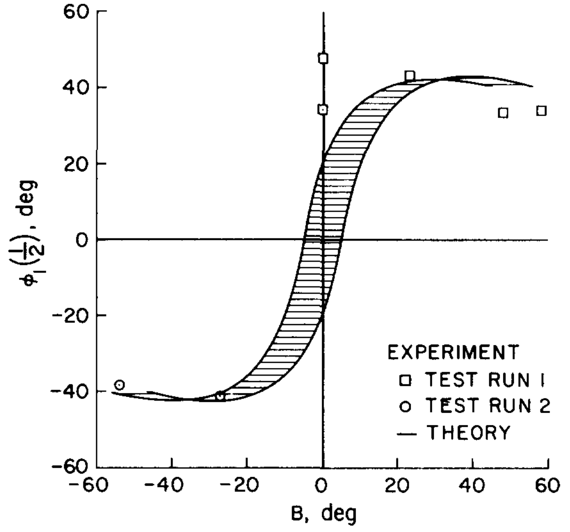


FIG. 9. Mid-length rotation for varying initial rigid-body rotations (nondimensional intensity $\gamma \approx 1{\cdot}32$).

sensitive the material is to temperature, the lower the critical intensity. It was found that the modulus of elasticity does not affect the critical intensity or beam twisting due to thermal loading.

The effects of initial imperfections were investigated, including both linear-twist and rigid-body-type imperfections. It was found that, at constant heat intensity, the twist response will not always continue to increase with increasing initial imperfection (see, e.g. Fig. 9), although it will continue for sufficiently low intensities.

It is interesting to note that no relative rotation of the ends can occur when the beam ends are not restrained from warping, regardless of the thermal loading.

From the correlation of theoretical and experimental results, one may conclude that the analysis is satisfactory for the prediction of torsional stability and deformation of tubes with open circular cross section exposed to thermal radiation loading. However, this is only one of a large class of structural elements which may be used singly or in combination to form structures that will be exposed to radiation loading (space-craft antennae, for example). Therefore, each of these structures must be similarly analyzed to insure structural integrity.

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APPENDIX

Thermal loading

Define the quantity on the left-hand side of equation (28) to be $G(\phi)$; therefore,

$$G(\phi) = \alpha E \int_0^m [\omega_s(m) - \omega_s(s)] \frac{\partial \bar{T}(s, z)}{\partial z} t(s)\, ds. \tag{A.1}$$

The purpose of this appendix is to show that, for the circular cross section (Fig. 3)

$$G(\phi) = -\frac{\alpha E R^5 Q}{K} B[\phi(z)]\phi'(z)$$

where $B(\phi)$ is defined by equations (29) and (30).

For the circular cross section

$$r(\theta) = R(1+\varepsilon \cos \theta) \tag{A.2}$$

$$\omega_s(\theta) = \int_0^s r(\bar{s})\, d\bar{s} = \int_0^\theta R(1+\varepsilon \cos \bar{\theta}) R\, d\bar{\theta}$$

$$\omega_s(\theta) = R^2(\theta + \varepsilon \sin \theta) \tag{A.3}$$

and

$$\omega_s(m) = \omega_s(2\pi) = 2\pi R^2. \tag{A.4}$$

If the quantity ψ is defined

$$\psi = \theta - \pi \tag{A.5}$$

then from equations (A.3) and (A.4)

$$\omega_s(m) - \omega_s(\psi) = R^2(\pi - \psi + \varepsilon \sin \psi). \tag{A.6}$$

If the twist rates of the beam, $\phi'(z)$, are sufficiently small, the thermal conduction in the axial, z, direction can be neglected. The temperature distribution at any element depends only on the rotation of the section relative to the source, $\phi(z)$, and the circumferential coordinate, s, of the element

$$T(s, z) = T[s, \phi(z)] = T[\psi, \phi(z)].. \tag{A.7}$$

The function of temperature in the function $G(\phi)$ [equation (A.1)] can be rewritten†

$$\frac{\partial \bar{T}(s, z)}{\partial z} = \frac{\partial \bar{T}(s, \phi(z))}{\partial \phi} \frac{\partial \phi(z)}{\partial z} = \frac{\partial \bar{T}(\psi, \phi)}{\partial \phi} \phi'(z). \tag{A.8}$$

† Since $T(s, z) = \bar{T}(s, z) + T_0(z)$ and T_0 is a constant for input flux uniform along the length, z, then $[\partial \bar{T}(s, z)]/\partial z = [\partial T(s, z)]/\partial z$.

Equations (A.1), (A.6) and (A.8) can be combined to obtain

$$G(\phi) = \alpha ER^3 t\phi'(z) \int_{-\pi}^{\pi} (\pi - \psi + \varepsilon \sin\psi) \frac{\partial \bar{T}(\psi, \phi)}{\partial \phi} \, d\psi.$$

Since the integral of the net flux does not change when there is a change in ϕ, the average temperature is independent of ϕ or

$$\int_{-\pi}^{\pi} \frac{\partial \bar{T}(\psi, \phi)}{\partial \phi} \, d\psi = 0$$

and $G(\phi)$ reduces to

$$G(\phi) = \alpha ER^3 t\phi'(z) \int_{-\pi}^{\pi} (-\psi + \varepsilon \sin\psi) \frac{\partial \bar{T}(\psi, \phi)}{\partial \phi} \, d\psi. \tag{A.9}$$

The applied thermal flux on an element of the beam is

$$Q\delta(\theta, \phi) \cos(\theta + \phi)$$

where $\delta(\theta, \phi)$ is a delta function (1 or 0) and the flux is shown schematically in Fig. 10. The total flux rate into an element is

$$Q\delta(\theta, \phi) \cos(\theta + \phi) R \, d\theta - \frac{Kt}{R} \frac{\partial T(\theta, \phi)}{\partial \theta}$$

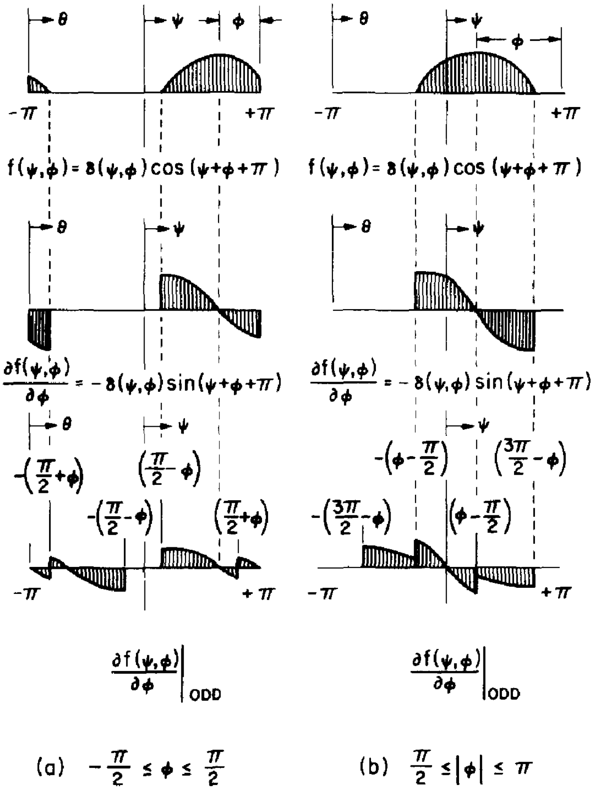

FIG. 10. The functions $f(\psi, \phi)$, $\partial f(\psi, \phi)/\partial\phi$ and $\partial f(\psi, \phi)/\partial\phi|_{\text{odd}}$.

and the total flux rate out of an element is

$$-\frac{Kt}{R}\frac{\partial T(\theta,\phi)}{\partial\theta}-\frac{Kt}{R}\frac{\partial^2 T(\theta,\phi)}{\partial\theta^2}\,\mathrm{d}\theta+\sigma\varepsilon_T T^4(\theta,\phi)R\,\mathrm{d}\theta.$$

The heat balance equation becomes

$$Kt\frac{\partial^2 T(\theta,\phi)}{\partial\theta^2}-R^2\sigma\varepsilon_T T^4(\theta,\phi) = -QR^2\delta(\theta,\phi)\cos(\theta+\phi) \tag{A.10}$$

where σ is the Stefan–Boltzmann constant and ε_T the emissivity of the beam surface.

If the thermal conductivity, K, is sufficiently large the variation of the temperature, $\bar{T}(\theta,\phi)$, is small when compared to the average absolute temperature, T_0; thus

$$T(\theta,\phi) = T_0+\bar{T}(\theta,\phi), \qquad |\bar{T}(\theta,\phi)| \ll T_0. \tag{A.11}$$

This feature is designed into many structures subjected to thermal loading in order to reduce bending deformations. Introduction of relation (A.11) into equation (A.10) and elimination of higher order terms produces

$$\frac{\partial^2\bar{T}(\theta,\phi)}{\partial\theta^2}-4\frac{\sigma\varepsilon_T T_0^3R^2}{Kt}\bar{T}(\theta,\phi) = \frac{\sigma\varepsilon_T T_0^4R^2}{Kt}-\frac{QR^2}{Kt}\delta(\theta,\phi)\cos(\theta+\phi). \tag{A.12}$$

In the design range of many structures

$$\frac{\sigma\varepsilon_T T_0^3R^2}{Kt} \ll 1 \tag{A.13}$$

thus equation (A.12) can be reduced to

$$\frac{\partial^2\bar{T}(\theta,\phi)}{\partial\theta^2} = \frac{\sigma\varepsilon_T T_0^4R^2}{Kt}-\frac{QR^2}{Kt}\delta(\theta,\phi)\cos(\theta+\phi)$$

or in terms of the variable ψ

$$\frac{\partial^2\bar{T}(\psi,\phi)}{\partial\psi^2} = \frac{\sigma\varepsilon_T T_0^4R^2}{Kt}-\frac{QR^2}{Kt}\delta(\psi,\phi)\cos(\psi+\phi+\pi). \tag{A.14}$$

If there is no flux through the boundaries $\psi = \pm\pi$ then

$$\frac{\partial\bar{T}(\psi,\phi)}{\partial\psi} = 0 \quad \text{at } \psi = \pm\pi. \tag{A.15}$$

Let

$$f(\psi,\phi) = \delta(\psi,\phi)\cos(\psi+\phi+\pi). \tag{A.16}$$

If equation (A.14) is integrated twice with respect to ψ and boundary condition (A.15) applied, it can be shown that

$$\bar{T}(\psi,\phi) = \left[\frac{QR^2}{2\pi Kt}\int_{-\pi}^{\pi} f(\bar{\psi},\phi)\,\mathrm{d}\bar{\psi}\right](\tfrac{1}{2}\psi^2+\pi\psi)-\frac{QR^2}{Kt}\int_0^{\psi}\int_{-\pi}^{\bar{\psi}} f(\bar{\bar{\psi}},\phi)\,\mathrm{d}\bar{\bar{\psi}}\,\mathrm{d}\bar{\psi}+C_1(\phi) \tag{A.17}$$

and

$$\frac{\partial\bar{T}(\psi,\phi)}{\partial\phi} = \left[\frac{QR^2}{2\pi Kt}\int_{-\pi}^{\pi}\frac{\partial f(\bar{\psi},\phi)}{\partial\phi}\,\mathrm{d}\bar{\psi}\right](\tfrac{1}{2}\psi^2+\pi\psi)-\frac{QR^2}{Kt}\int_0^{\psi}\int_{-\pi}^{\bar{\psi}}\frac{\partial f(\bar{\bar{\psi}},\phi)}{\partial\phi}\,\mathrm{d}\bar{\bar{\psi}}\,\mathrm{d}\bar{\psi}+\frac{\partial C_1(\phi)}{\partial\phi}$$

or

$$\frac{\partial \overline{T}(\psi,\phi)}{\partial \phi} = \frac{QR^2}{2\pi Kt}\int_{-\pi}^{\pi}\frac{\partial f(\overline{\psi},\phi)}{\partial \phi}\,\mathrm{d}\overline{\psi}(\tfrac{1}{2}\psi^2) + \frac{QR^2}{2Kt}\left[\int_0^{\pi}\frac{\partial f(\overline{\psi},\phi)}{\partial \phi}\,\mathrm{d}\overline{\psi} - \int_{-\pi}^{0}\frac{\partial f(\overline{\psi},\phi)}{\partial \phi}\,\mathrm{d}\overline{\psi}\right]\psi$$

$$-\frac{QR^2}{Kt}\int_0^{\psi}\int_0^{\overline{\psi}}\frac{\partial f(\overline{\overline{\psi}},\phi)}{\partial \phi}\,\mathrm{d}\overline{\overline{\psi}}\,\mathrm{d}\overline{\psi} + \frac{\partial C_1(\phi)}{\partial \phi}. \tag{A.18}$$

Only functions $\partial \overline{T}(\psi,\phi)/\partial\phi$ odd in ψ [defined $\partial \overline{T}(\psi,\phi)/\partial\phi|_{\text{odd}}$] will contribute to the function $G(\phi)$, equation (A.9). The first and last terms on the right-hand side of equation (A.18) are even in ψ and do not contribute to $\partial \overline{T}(\psi,\phi)/\partial\phi|_{\text{odd}}$. The second term contributes only from $\partial f(\psi,\phi)/\partial\phi$, which is odd in ψ [denoted $\partial f(\psi,\phi)/\partial\phi|_{\text{odd}}$] and reduces to

$$\psi\frac{QR^2}{Kt}\int_0^{\pi}\left.\frac{\partial f(\overline{\psi},\phi)}{\partial \phi}\right|_{\text{odd}}\mathrm{d}\overline{\psi}.$$

The third term also contributes only for $\partial f(\psi,\phi)/\partial\phi|_{\text{odd}}$, thus

$$\left.\frac{\partial \overline{T}(\psi,\phi)}{\partial \phi}\right|_{\text{odd}} = \frac{QR^2}{Kt}\left\{\psi\int_0^{\pi}\left.\frac{\partial f(\overline{\psi},\phi)}{\partial \phi}\right|_{\text{odd}}\mathrm{d}\overline{\psi} - \int_0^{\psi}\int_0^{\overline{\psi}}\left.\frac{\partial f(\overline{\overline{\psi}},\phi)}{\partial \phi}\right|_{\text{odd}}\mathrm{d}\overline{\overline{\psi}}\,\mathrm{d}\overline{\psi}\right\}. \tag{A.19}$$

The function $G(\phi)$, equation (A.9) becomes

$$G(\phi) = -\frac{\alpha E R^5 Q}{K}\phi'(z)B[\phi(z)] \tag{A.20}$$

where

$$B[\phi(z)] = -2\int_0^{\pi}(\psi - \varepsilon\sin\psi)\left[\int_0^{\psi}\int_0^{\overline{\psi}}\left.\frac{\partial f(\overline{\overline{\psi}},\phi)}{\partial \phi}\right|_{\text{odd}}\mathrm{d}\overline{\overline{\psi}}\,\mathrm{d}\overline{\psi} - \psi\int_0^{\pi}\left.\frac{\partial f(\overline{\psi},\phi)}{\partial \phi}\right|_{\text{odd}}\mathrm{d}\overline{\psi}\right]\mathrm{d}\psi. \tag{A.21}$$

The function $\partial f(\psi,\phi)/\partial\phi|_{\text{odd}}$, shown graphically in Fig. 10, is for $-\pi/2 \le \phi \le \pi/2$:

TABLE 2

ψ			$\partial f(\psi,\phi)/\partial\phi\vert_{\text{odd}}$
$-\pi$	to	$-\left(\frac{\pi}{2}+\phi\right)$	$-\frac{1}{2}\sin(\psi+\phi+\pi)-\frac{1}{2}\sin(\psi-\phi-\pi)$
$-\left(\frac{\pi}{2}+\phi\right)$	to	$-\left(\frac{\pi}{2}-\phi\right)$	$-\frac{1}{2}\sin(\psi-\phi-\pi)$
$-\left(\frac{\pi}{2}-\phi\right)$	to	$\left(\frac{\pi}{2}-\phi\right)$	0
$\left(\frac{\pi}{2}-\phi\right)$	to	$\left(\frac{\pi}{2}+\phi\right)$	$-\frac{1}{2}\sin(\psi+\phi+\pi)$
$\left(\frac{\pi}{2}+\phi\right)$	to	π	$-\frac{1}{2}\sin(\psi+\phi+\pi)-\frac{1}{2}\sin(\psi-\phi-\pi)$

and for $\pi/2 \le |\phi| \le \pi$:

TABLE 3

ψ			$\partial f(\psi, \phi)/\partial\phi\|_{\text{odd}}$
$-\pi$	to	$-\left(\frac{3\pi}{2}-\phi\right)$	0
$-\left(\frac{3\pi}{2}-\phi\right)$	to	$-\left(\phi-\frac{\pi}{2}\right)$	$-\frac{1}{2}\sin(\psi-\phi-\pi)$
$-\left(\phi-\frac{\pi}{2}\right)$	to	$\left(\phi-\frac{\pi}{2}\right)$	$-\frac{1}{2}\sin(\psi+\phi+\pi)-\frac{1}{2}\sin(\psi-\phi-\pi)$
$\left(\phi-\frac{\pi}{2}\right)$	to	$\left(\frac{3\pi}{2}-\phi\right)$	$-\frac{1}{2}\sin(\psi+\phi+\pi)$
$\left(\frac{3\pi}{2}-\phi\right)$	to	π	0

After substitution of $\partial f(\psi, \phi)/\partial\phi|_{\text{odd}}$ into equation (A.21) and much tedious but straightforward integration, it can be shown that $B(\phi)$ is as shown in equations (29) and (30).

(*Received* 18 *February* 1972; *revised* 15 *June* 1972)

Абстракт—Исследуется статическое крутильное равновесие злементов конструкции, обладающих открытым круглым поперечным сечением, подверженных действию, зависимой от деформации, термической нагрузки. Определяются уравнения равновесия и граничные условия. Установливаются критерия для малых деформаций и получаются формы равновесия для больших деформаций /в закритической области/. Представляются эффекты начальной неточности и получаются решения.

Сравниваются критерия устойчивости с решениями предыдущих исследователей. Указано что дефекты теории предыдущих исследований ведут к неправильным критериям. Предлагаемая теория находится в хорошом согласии с экспериментальными данными.